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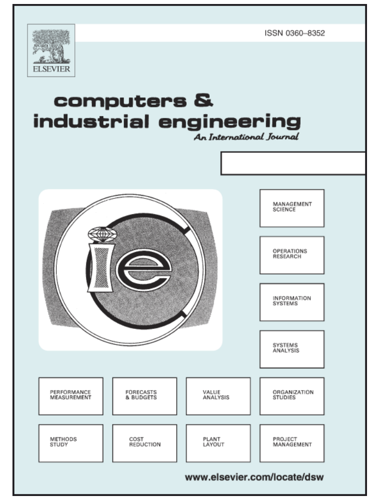

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A systematic warranty-reliability-price decision model for two-dimensional warranted products with heterogeneous usage rates

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- A warranty-reliability-price decision model for 2D warranted products is proposed.
- Heterogeneous usage rates of customers are considered in the model.
- A new sales rate function of 2D warranted product is proposed.
- Both a stable market condition and a dynamic market condition are investigated.
- It shows that the optimal nominal usage rate of 2D warranty region stays constant.

A systematic warranty-reliability-price decision model for two-dimensional warranted products with heterogeneous usage rates

Abstract

For products with a two-dimensional (2D) warranty policy, customers are heterogeneous on usage rates. In this study, we propose a systematic warranty-reliability-price combination decision model for repairable products with a 2D Free Replacement-Repair Warranty (FRW). The effects of customers' heterogeneous usage rate on both reliability design and the demand function of 2D warranted products are considered in the model. The optimal warranty-reliability-price combination is obtained by maximizing total expected profits over products' life cycle. Two marketing scenarios are discussed, i.e., a stable market and a dynamic market. The optimality conditions for the two scenarios are investigated. The applicability of the model is demonstrated via a case study of an industrial 3D printer manufacturer, and it shows the consistency of the model.

Keywords: two-dimensional warranty; price; reliability; heterogeneous usage rate; dynamic market

1. Introduction

The role of warranty between the manufacturer and the customer has become increasingly important, both as a protectional tool and as a promotional tool (Murthy and Djamaludin, 2002). As a protectional tool, the manufacturer is obligated to repair the failures during the warranty period free of charge. Warranty cost roughly ranges from 2 % to 15 % of net sales, which mainly depends on warranty terms and product's reliability. Optimal warranty and reliability design should be considered jointly (Murthy, 2006). Better warranty terms usually imply higher product's quality and require higher reliability design. Meanwhile, as a promotional tool, an attractiveness warranty policy is usually offered to signing it's quality and to promote products' sales in the marketplace. To launch new products successfully, sale price, warranty period and reliability parameters are the three key decisions for the manufacturer to determine. A well designed warranty-reliability-price combination usually helps the manufacturer to have a competitive advantage in the marketplace (Murthy and Blischke, 2000).

The sale price and warranty period are two key factors influencing the manufacturing' profit and customers' purchasing decisions in the marketplace. For base warranty (BW) and sale price optimization, Glickman and Berger (1976) firstly investigated the dependence of optimal profit on price and one-dimensional (1D) warranty period based on a log-linear demand function. Following the log-linear demand function proposed by Glickman and Berger (1976), there are considerable papers considering the price-warranty optimization for both a stable market and a dynamic market (Teng and Thompson, 1996; Lin and Shue, 2005; Zhou et al., 2009; Wu et al., 2006, 2009). Under different repair options, Matis et al. (2008) presented the optimal price and pro-rate warranty period. Lei et al. (2017) considered the problem of warranty price affecting customers' beliefs on product reliability, and the multi-period optimal dynamic product's sale price and warranty period were investigated. Shang et al. (2018) proposed a condition-based renewable replacement warranty policy, and the optimal warranty period, sale price, and replacement threshold were optimized in a monopoly market. Chien et al. (2020) studied the optimal price and warranty coverage assuming that the price was a linear function of the warranty period for both the renewing free-replacement warranty and the free minimal repair warranty. Fang (2020) proposed a rational non-cooperative game model to determine the optimal price and warranty in a competitive duopoly market.

However, the above literatures mainly focus on one-dimensional warranted products. Complex repairable new products, such as vehicles and large manufacturing equipment, are usually sold with a two-dimensional (2D) warranty region of the age limit and the usage limit. For example, new vehicles are often sold with a 2D warranty region of 3 years and $36,000$ miles. Xie (2017) investigated the optimal product's sale price and 2D base warranty region, and the Glickman-Berger model was extended for 2D warranted products. Considering customers' different perceptions on the 2D warranty period, a new demand function of 2D warranty products based on attractiveness index was proposed by He et al. (2017), and the optimal 2D base warranty period was investigated. To trade off the warranty cost and the boosted demand, Zhang et al. (2019) studied the optimal 2D warranty period design for three different cases considering the relationship between the nominal usage rate and customers' usage rates. For the multiple products sold through both online and offline channels, Taleizadeh and Mokhtarzadeh (2020) proposed an integrated model to determine the optimal product price for each channel and 2D warranty period considering the covariance among different components of products. Cheong et al. (2021) studied the joint dynamic optimization of product's price and 2D warranty based on a new sales function, while the influence of the different customers' usage rate was considered in the model.

In addition to base warranties, the optimal design of extended warranties (EW) was studied by many researchers. Considering the extended warranty with trade-in service, Bian et al. (2019) investigated the optimal EW length and the associated price. Hooti et al. (2020) investigated the optimal extended warranty length with the condition of limited number of minimal repairs during warranty period. A warranty model with a post-purchase option open to customers was designed by Liu et al. (2020), and the price of complimentary extended warranty was studied considering customers' heterogeneous risk attitudes. Wang et al. (2020) studied an extended warranty menu with multiply options of lengths and prices based on multinomial logit choice model, and the optimal design and price of the EW menu were investigated. For the 2D extended warranty, Zhang and Su (2019) derived the prices of 2D customized extended warranties when conducting four difference imperfect preventive maintenance strategy for three different usage rate customers. He et al. (2020) proposed a win-win price decision model considering the purchasing time of EW, customer's preventive maintenance options, and different usage rates from the perspective of both the manufacturer and the customers.

Meanwhile, the design of BW and EW can be optimized jointly. The optimal dynamic pricing of product price and the price of two-dimensional EW were derived by Wang et al. (2021). In the present model, the effects of repair learning characteristic, discrete preventive maintenance during the base warranty and extended warranty, and fluctuating product demand and EW demand in the market on dealers' profit were investigated. Mitra (2019) formulated a model considering that customers are price-sensitive to extended warranty, and the optimal product price, two-dimensional (2D) base warranty period, the 2D extended warranty period, and the price of EW were investigated. Afsahi and Shafiee (2020) proposed a stochastic simulation-optimization model for BW and EW, and an integrated approach was developed to optimize the base and extended warranty lengths, products' sale price, price of EW, repair strategy and the spare part inventory control policy for under and out-of-warranty products simultaneously.

The above literatures on warranty and price optimization for both BW and EW are studied for a given design reliability. The optimal decisions should be considered the relationships between engineering aspects (such as design reliability) and marketing aspects (such as sale price and warranty). The optimal warranty-reliability-price combination in an oligopoly was firstly investigated by DeCroix (1999). For a free replacement-repair warranty (FRW), Huang et al. (2007) proposed an integrated model to optimize the 1D base warranty length, products' sale price, and the design reliability jointly for both a stable market and a dynamic market. Darghouth et al. (2017) investigated the optimal 1D base warranty period, product reliability and sale price considering four different maintenance service contract options in warranty and/or post-warranty period. Zhu et al. (2019) considered the effects of sales promotion on product reliability design and warranty policy decisions, and reliability-related sales growth models for regular sales and promotions were proposed. The optimal 1D base warranty length, product reliability, regular price, promotion price and lengths of regular sales and promotions were investigated by an integrated optimization model. The above literatures mainly focus on 1D warranty-price-reliability jointly optimization.

A brief summary of the literature on warranty optimization and the research gaps this paper seeks to fill are given in Table 1. This study attempts to propose a systematic warranty-reliability-price combination decision model for 2D warranted products with heterogeneous usage rates, and two marketing scenarios, i.e., a stable market condition and a dynamic market condition, are investigated. The main contributions of this study are given as follows. Firstly, we extend base warranty, design reliability, and sales price combination decision model for 2D warranted products considering the effects of customers' heterogeneous usage rate on both reliability design and the demand function of 2D warranted products. Secondly, the effects of heterogeneous usage rates on demand of 2D warranted products in a dynamic market are studied considering the diffusion effects, which differs from related demand models in the static market (Manna, 2008; He et al., 2017). A new sales function of 2D warranted product considering customers' heterogeneous usage rate is proposed based on the mean coverage of 2D warranty period, and the impacts of market adoption and diffusion on demand is considered. A longer warranty coverage means more failures can be repaired free of charge by the manufacturer, which means the 2D warranty is more attractive to the customers. Lastly, some different findings are derived when considering 2D warranty-reliability-price combination jointly. Compared with 1D warranty-price-reliability literatures (Huang et al., 2007; Zhu et al., 2019; Darghouth et al., 2017), we found that the optimal nominal usage rate of 2D warranty period stays constant, and the optimal price and 2D warranty period decrease or increase simultaneously in the dynamic market condition. Differing from the work by Cheong et al. (2021), which studied the joint dynamic optimization of product's price and 2D warranty, they concluded that the dynamic decision making was better than static decision making in achieving higher profit. However, when considering the design reliability jointly with price and warranty, we found that the expected profit in the static market was higher than that in the dynamic market, but the market share of a dynamic market condition was higher than that of a stable market condition. That's because the design reliability of products in the static market and the dynamic market are same in the work by Cheong et al. (2021). When considering the design reliability jointly with price and warranty, the optimal design reliability of conducting a dynamic market strategy is greater than that of conducting a stable market strategy.

Table 1: Related literature on warranty optimization

	Product Price		BW [a] Period		EW [b]		Reliablity
	Static	Dynamic	1D [c]	2D [d]	Price	Length	
Zhou et al. (2009)	✓	–	✓	–	–	–	–
Shang et al. (2018)	✓	–	✓	–	–	–	–
Fang (2020)	✓	–	✓	–	–	–	–
Hooti et al. (2020)	–	–	✓	–	–	–	–
Lei et al. (2017)	–	✓	✓	–	–	–	–
Chien et al. (2020)	–	✓	✓	–	–	–	–
Bian et al. (2019)	–	–	–	–	✓	✓	–
Liu et al. (2020)	–	–	–	–	✓	✓	–
Wang et al. (2020)	–	–	–	–	✓	✓	–
Wang et al. (2021)	✓	✓	–	–	✓	–	–
Mitra (2019)	✓	–	–	✓	✓	✓	–
Afsahi and Shafiee (2020)	–	✓	✓	–	✓	✓	–
Huang et al. (2007)	✓	✓	✓	–	–	–	✓
Zhu et al. (2019)	–	✓	✓	–	–	–	✓
Darghouth et al. (2017)	✓	✓	✓	–	–	–	✓
He et al. (2017)	–	–	–	✓	–	–	–
Zhang et al. (2019)	–	–	–	✓	–	–	–
Xie (2017)	✓	–	–	✓	–	–	–
Taleizadeh and Mokhtarzadeh (2020)	✓	–	–	✓	–	–	–
Cheong et al. (2021)	✓	✓	–	✓	–	–	–
This Paper	✓	✓	–	✓	–	–	✓

[a] Base Warranty. [b] Extended Warranty. [c] One-dimensional Warranty. [d] Two-dimensional Warranty.

The remainder of this paper is organized as follows. Section 2 is the problem description. Section 3 introduces engineering related models and markerting decision models. Section 4 gives the optimal conditions for stable market and dynamic market. Section 5 presents a case study. Section 6 gives the conclusions and future works.

2. Problem description

We consider a manufacturer provides a repairable product to the marketplace, which deteriorates at both time scale and usage scale. A two-dimensional (2D) Free Replacement-Repair Warranty (FRW) policy $\Omega(L, M)$ is offered, under which the manufacturer agrees to repair or provide replacements for failed items free of charge during warranty period $[0, L] \times [0, M]$. L and M are the age limit and the usage limit of base warranty period, respectively. $r_0 = \frac{M}{L}$ is the nominal usage rate of the 2D warranty policy. Meanwhile, customers in the market are usually heterogeneous in usage rates, and usage rate R for the customer population is a random variable. The probability density function of R is $g(r)$ over $[r_l, r_u]$, where r_l and r_u are the minimal usage rate and the maximum usage rate, respectively. $g(r)$ can be estimated by the manufacturer based on historical warranty data or maintenance records (Lawless et al., 2009). The product's life cycle in the market is T, and the manufacturer will stop to providing the product on the marketplace when sales time $t > T$.

The reliability parameter θ is a design parameter of the product's reliability $R(t|\theta)$. To be consistent with previous literatures (Huang et al., 2007; Darghouth et al., 2017), we use the reliability parameter θ to indicate the product's reliability $R(t|\theta)$, and it is determined at the design stage. Let $f(t|\theta)$ be the probability density function of product failure times, and $h(t|\theta)$ be the hazard function. For example, for the exponential distribution, the design parameter θ is the parameter of the exponential distribution, and $f(t|\theta) = \theta \exp(-\theta t)$, $h(t|\theta) = \theta$, and $R(t|\theta) = \exp(-\theta t)$. For the Weibull distribution, the design parameter θ is the scale parameter of the distribution, and β is the shape parameter. In this case, $f(t|\theta) = \theta^{\beta} \beta t^{\beta-1} \exp(-\theta t)^{\beta}$, and $h(t|\theta) = \beta \theta^{\beta} t^{\beta-1}$. For the convenience of discussion, we assume that $\theta_l \leq \theta \leq \theta_u$, and a smaller value of θ implies a small value of hazard function

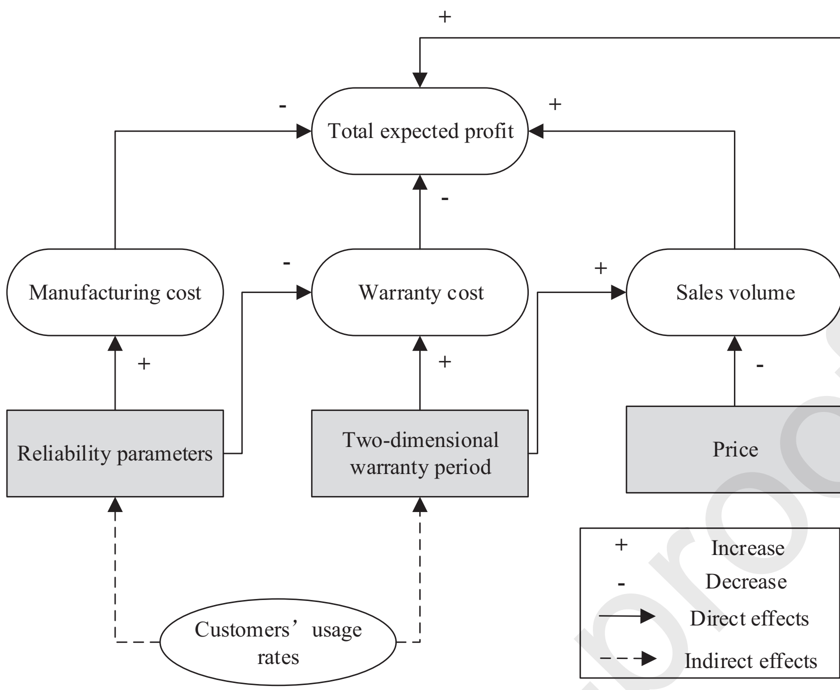

Figure 1: Influence diagram of interactions among price, reliability parameter, and 2D warranty period.

$h(t|\theta)$, and vice versa. Because of limits of technology and budgets for the manufacturer, the design parameter θ has a minimum value θ_l, which represents the upper bound of achievable reliability. θ_u is the maximum value and indicates the lower bound of product reliability required for product's satisfactory operation. The product's design reliability and related parameters can be estimated based on reliability data or related warranty data (Jung and Bai, 2007; Lawless et al., 2009).

The relationships among price, warranty period and reliability parameter design are shown in Fig. 1. The problem of the manufacturer is how to determine the optimal 2D period $\Omega(L, M)$, price P and reliability parameter θ to maximize the total expected discounted profit over life cycle T. Specifically, the following two scenarios of marketing conditions for price and 2D warranty period are considered.

(a) Scenario 1: Sale price P and the two-dimensional warranty period $\Omega(L, M)$ are constant over the product's life cycle T, which is the stable market condition.

(b) Scenario 2: Both sale price and the two-dimensional warranty period change over the product's life cycle T, which is the dynamic market condition.

Since θ is determined at the design stage before launching to the marketplace, the designed reliability parameter θ is constant for the above two scenarios.

The following main assumptions are considered.

(1) Assume that the usage rate r of a customer stays constant during the warranty period, but it is a random variable for customers population. This assumption can be validated by related literatures on two-dimensional warranty data analysis (Jung and Bai, 2007; Lawless et al., 2009).

(2) Historical warranty data or/and maintenance data of similar products are available for the manufacturer to estimate the distribution of customers' usage rate (Lawless et al., 2009).

(3) All failures during 2D warranty period are minimally repaired by the manufacturer, and the repair time is negligible compared to warranty period (Baik et al., 2004).

The mathematical notations are summarized in Table 2.

3. Model formulation

3.1. The reliability of the products considering usage rates

For products with a 2D warranty period, they usually deteriorate at both time t and cumulative usage u. To model the relation between failure rate and the usage rate, there are two rather obvious approaches, which are proportional hazard (PH) approach and accelerated failure time (AFT) approach (Lawless et al., 1995). The PH approach is employed in this study, where the hazard function $h(t|r,\theta)$ for the products with usage rate r is given as Equation 1.

$$h(t|r,\theta) = h_0(t|\theta)\Phi(r), \tag{1}$$

Table 2: Notations and description

Notation	Description
L, M	the age limit and usage limit of 2D warranty period, respectively (*decision variable*)
P	the unit sale price (*decision variable*)
θ	the design reliability parameter (*decision variable*)
r_0	$r_0 = M/L$, which is the nominal usage rate of the 2D warranty period
$h(t\|r, \theta)$	the hazard function for customers with usage rate r
$C_d(\theta)$	the design and development cost
$C_m(t, \theta)$	the production cost per unit at sales time t
$C(t, \theta)$	the total cost per unit at sales time t
C_r	the average minimal repair cost per unit during warranty period
$W(L, M\|\theta)$	the expected warranty cost of 2D warranty period $\Omega(L, M)$
$Q(t)$	the cumulative sales quantity at sales time t
$q(t)$	$q(t) = \frac{dQ(t)}{dt}$,which is the sales rate of products at sales time t
i	the annual rate of discounting
J	the total expected discounted profit

where $h_0(t|\theta)$ is the baseline hazard function under the designed usage rate, and $\Phi(r)$ is a positive-valued function to modelling the effects on different usage rate r on products' reliability. θ is the design parameter determined by the manufacturer, and $\theta_l \leq \theta \leq \theta_u$.

3.2. The design cost and production cost

Before launching to the marketplace, the design parameter θ is determined at the design and development stage. In general, the design and development cost $C_d(\theta)$ is usually a monotonically increasing function of product reliability. An exponential relationship between $C_d(\theta)$ and θ is referred by Huang et al. (2007) which is modelled in Equation 2.

$$C_d(\theta) = A_1 + B_1 \exp\left(k\frac{\theta_u - \theta}{\theta - \theta_l}\right), \tag{2}$$

where $A_1 > 0$ is the fixed design and development cost, and $B_1 > 0$ represents the cost of improving product's reliability. $0 < k \leq 1$ represents the difficulties of improvement in reliability for the product, which depends on the product's complexity, technological and resource limitations, etc. According to Equation 2, when design parameter θ decreases (the product's reliability increases), the design and development cost $C_d(\theta)$ increases.

Meanwhile, the reliability parameter θ also influences products' unit manufacturing cost. As discussed by Huang et al. (2007), the initial manufacturing cost has the form as Equation 3.

$$C_{m0} = A_2 + B_2\theta^{-\gamma_1}, \tag{3}$$

where $A_2 > 0$ is the fixed unit manufacturing cost parameter, and $B_2 > 0$ is a reliability related parameter. $\gamma_1 > 0$ is a positive-value parameter. As shown in Equation 3, a decrease of θ leads to an increase of the initial manufacturing cost.

The learning effect of production for the manufacturer is also considered. With the increasing number of total sales volume, the unit manufacturing cost will decrease due to the proficiency in production. Considering the learning curve of production, the production cost per item is given as Equation 4.

$$\begin{aligned} C_m(t,\theta) &= KC_{m0}\left[\frac{Q_0}{Q(t)}\right]^{\gamma_2} \\ &= K(A_2 + B_2\theta^{-\gamma_1})\left[\frac{Q_0}{Q(t)}\right]^{\gamma_2}, \end{aligned} \tag{4}$$

where $Q_0 = Q(0)$ and $Q(t)$ represent the cumulative sales quantity at sales time $t = 0$ and t, respectively. $0 < \gamma_2 < 1$ is the learning parameter for the production process, and is assumed as a constant. $K > 0$ is a positive-value parameter, which represents the influence of other various factors such as production rate and inflation. As described by Equation 4, with the increasing of cumulative sales quantity $Q(t)$, the production cost per item $C_m(t,\theta)$ decreases due to the learning effect.

Therefore, during the design stage and production stage, the total cost per unit at sales time t is given as Equation 5.

$$\begin{aligned} C(t,\theta) &= \frac{C_d(\theta)}{Q(T)} + C_m(t,\theta) \\ &= \frac{A_1 + B_1 \exp\left(k\frac{\theta_u-\theta}{\theta-\theta_l}\right)}{Q(T)} + K(A_2 + B_2\theta^{-\gamma_1})\left[\frac{Q_0}{Q(t)}\right]^{\gamma_2}, \end{aligned} \tag{5}$$

where $Q(T)$ is the total sales volume over the life cycle $[0, T]$.

3.3. Warranty cost analysis

For products with a 2D warranty period $\Omega(L, M)$, $r_0 = M/L$ is the nominal usage rate of the 2D warranty period. Assume that all the failures during warranty period are minimal repaired by the manufacturer. Using minimal repair by which the failed components are repaired or replaced, and the failure rate of the product stays unchanged after rectifications (Barlow and Hunter, 1960). For customers with usage rate r, the failure time follows non-homogeneous Poisson process (NHPP) with intensity function $\lambda(t|r) = h(t|r)$ (Manna et al., 2008). The expected number of warranty claims over interval $[0, t)$ is given as Equation 6.

$$E[N(t|r)] = \Lambda(t|\theta) = \int_0^t h(t|r,\theta)dt. \tag{6}$$

Customers are heterogeneous on usage rate during warranty period. For customers with usage rate $r_l \leq r \leq r_0$, they belong to the low-usage type customers, and their expiration of warranty period is L at time and rL at usage. Similarly, for customers with usage rate $r_0 \leq r \leq r_u$, they are high-usage type customers with expiration warranty of M/r at time and M at usage. Therefore, for the customers population, the expected number of warranty claims during $\Omega(L, M)$ is given in Equation 7.

$$N(L, M|\theta) = \int_{r_l}^{r_0} \Lambda(L|\theta)g(r)dr + \int_{r_0}^{r_u} \Lambda(\frac{M}{r}|\theta)g(r)dr, \tag{7}$$

where $g(r)$ is the probability density function of usage rate. The expected warranty cost $W(L, M|\theta)$ of 2D warranty period $\Omega(L, M)$ is given as follows.

$$W(L, M|\theta) = C_r N(L, M|\theta), \tag{8}$$

where C_r is the average minimal repair cost per unit.

3.4. Sales rate function

For complex products with a 2D warranty period $\Omega(L, M)$, both warranty period and sale price are often used as a promotion marketing tool. The log-linear sales rate function proposed by Glickman and Berger (1976) has been widely used for products with one-dimensional warranty period, which is given as follows.

$$q(P, W_1) = k_1(k_2 + W_1)^{\psi_1} P^{-\psi_2}, \tag{9}$$

where W_1 is the warranty period of one-dimensional warranty. However, for products with a two-dimensional warranty, customers' various usage rates lead to different warranty expiration time and usage. Manna (2008) used a mean coverage time W_2 of 2D warranty period to replace W_1 in one-dimensional warranty, which is given in Equation 10.

$$W_2 = \int_{r_l}^{r_0} Lg(r)dr + \int_{r_0}^{r_u} \frac{M}{r} g(r)dr, \tag{10}$$

where W_2 is the customers' mean coverage time of 2D warranty period, and $g(r)$ is the probability density function of customers' usage rate.

However, the demand function proposed by Manna (2008) is a static model when price P and 2D warranty period $\Omega(L, M)$ are constant. In a dynamic market, P and 2D warranty period $\Omega(L, M)$ change over time, the impacts of market adoption and diffusion on demand function should be considered. One of the well-known first-purchase diffusion models of new product diffusion in marketing is the Bass's growth model (Bass, 1969). Based on the Bass's growth model, there are two basic kinds of purchasers in the market, which are the innovators and the imitators. Q_M is the maximum sales potential in the market. $Q(t)$ is the accumulated sales volume over $[0, t]$. The volume of sales to the innovators is proportional to the number of potential customers who do not already own the product, $Q_M - Q(t)$. The volume of sales to the imitators is proportional to the number of potential customers, $Q_M - Q(t)$, and the number of customers who do have the product $Q(t)$. Summing these two terms, it is rearranged and obtained as $\left[1 - \frac{Q(t)}{Q_M}\right]\left[\psi + \frac{Q(t)}{Q_M}\right]$, which reflects the concept of sales as a diffusion process involving innovators and imitators as in the Bass's growth model (Robinson and Lakhani, 1975).

We refer to the sales model presented by Manna (2008),Huang et al. (2007) and Darghouth et al. (2017), and considering customer's different usage rates, the sales rate for products with a 2D warranty period is given in Equation 11.

$$q(t) = k_1 \left[k_2 + \int_{r_l}^{r_0} Lg(r)dr + \int_{r_0}^{r_u} \frac{M}{r} g(r)dr\right]^{\psi_1} P^{-\psi_2} \left[1 - \frac{Q(t)}{Q_M}\right]\left[\psi + \frac{Q(t)}{Q_M}\right], \tag{11}$$

where $Q(t)$ is the accumulated sales volume over $[0, t]$, and $q(t) = \frac{dQ(t)}{dt}$. Specifically, $Q(t) = Q_0 + \int_0^t q(t)dt$, where $Q_0 = Q(0)$ is the initial sales quantity and can be estimated by historical sales data. $\psi > 0$ reflects the relative influence of innovators. As in Equation 11, sales rate function $q(t)$ can be viewed as a form of separable demand functions as discussed by Teng and Thompson (1996), where $q(t) = F(P, L, M)\, G(Q(t))$.

3.5. The total expected discounted profit

The total expected discounted profit over product's life cycle T mainly depends on: (1) the sale price P; (2) the development and design cost $C_d(\theta)$; (3) the production cost per unit $C_m(t, \theta)$; (4) the expected warranty cost $W(L, M|\theta)$, and (5) the sales rate of products with

2D warranty $q(t)$. In the following, two scenarios are discussed, including a stable market condition and a dynamic market condition.

3.5.1. A stable market condition (scenario 1)

When the manufacturer conducts a stable market strategy, both the sale price P and 2D warranty period $\Omega(L, M)$ are invariant over the life cycle T. Thus, the sales rate $q(t)$ of Equation11 can be written as $q(P, L, M)$, which is invariant over the life cycle T. The accumulated sales volume is $Q(T) = Q_0 + \int_0^t q(t)dt = Q_0 + q(P, L, M)T$. Therefore, the total expected discounted profit J is given in Equation 12.

$$J(P, L, M, \theta) = (P - C(T, \theta) - W(L, M|\theta))\, Q(T) \exp(-iT). \tag{12}$$

The optimal warranty-price-reliability combination in the stable market condition is given as in Equation 13.

$$\text{Max: } J(P, L, M, \theta) = (P - C(T, \theta) - W(L, M|\theta))\, Q(T) \exp(-iT). \tag{13}$$

subject to:

$$\begin{cases} Q(T) = Q_0 + qT \\ \theta_l \leq \theta \leq \theta_u \\ P \geq 0 \\ L \geq 0 \\ M \geq 0 \end{cases}.$$

$i \geq 0$ is the annual rate of discounting future profit. $q > 0$ is the sales rate of products with 2D warranty, which is a constant variable over the life cycle T when sale price P and $\Omega(L, M)$ are invariant.

3.5.2. A dynamic market condition (scenario 2)

When a dynamic market condition is conducted, the sale price $P(t)$ and 2D warranty period $\Omega(L(t), M(t))$ changes over the life cycle T. Thus, the total expected discounted profit J is given in Equation 14.

$$J = \int_0^T [P(t) - C(t, \theta) - W(L(t), M(t)|\theta)]q(t) \exp(-it)dt. \tag{14}$$

The optimal warranty-price-reliability combination in the dynamic market condition is given as in Equation 15

$$\text{Max: } J = \int_0^T [P(t) - C(t,\theta) - W(L(t), M(t)|\theta)]q(t)\exp(-it)dt. \tag{15}$$

subject to:

$$\begin{cases} q(t) = \dfrac{dQ(t)}{dt} \\ \theta_l \leq \theta \leq \theta_u \\ P(t) \geq 0, L(t) \geq 0, M(t) \geq 0 \end{cases}.$$

$i \geq 0$ is the annual rate of discounting future profit.

4. Model analysis

4.1. Optimality conditions for a stable market (scenario 1)

The problem of the manufacturer is to determine the optimal sale price P, 2D warranty period $\Omega(L, M)$ and reliability parameter θ to maximize the total expected discounted profit J. In the stable market, the necessary conditions for P^*, L^*,M^* and θ^* are given as follows.

$$\begin{cases} \dfrac{\partial J}{\partial P} = 0 \\ \dfrac{\partial J}{\partial L} = 0 \\ \dfrac{\partial J}{\partial M} = 0 \\ \dfrac{\partial J}{\partial \theta} = \tau \end{cases}. \tag{16}$$

Given that $\theta \in [\theta_l, \theta_u]$, the sign of the arbitrary constant τ is given as follows.

$$\begin{cases} \tau \leq 0, \text{if } \theta^* = \theta_l \\ \tau = 0, \text{if } \theta_l < \theta^* < \theta_u \\ \tau \geq 0, \text{if } \theta^* = \theta_u \end{cases}. \tag{17}$$

4.2. Optimality conditions for a dynamic market (scenario 2)

In the dynamic market, sale price $P(t)$ and 2D warranty period $\Omega(L(t), M(t))$ change over the life cycle $[0, T]$. In practice, the nominal usage rate $r_0 = \frac{M(t)}{L(t)}$ of 2D warranty region is constant over product' life cycle $[0, T]$. When r_0 is a constant variable during the life cycle, the 2D warranty period can be viewed as $\Omega(L(t), r_0 L(t))$.

To obtain the optimal solution, we apply the maximum principle to solve the dynamic optimization problem (Sethi, 2019). The current value of the Hamiltonian function is given in Equation 18.

$$H = [P(t) - C(t, \theta) - W(L(t), M(t), \theta) + \lambda(t)]q(t), \tag{18}$$

where $\lambda(t)$ is the current value adjoint variable. The current value of the Hamiltonian H can be interpreted as the instantaneous total profit at time t, and $\lambda(t)$ is interpreted as the future benefit of producing one more unit at time t. $\lambda(t)$ satisfy the following differential equation in Equation 19, which is satisfied with the transversality condition $\lambda(T) = 0$.

$$\begin{aligned} \dot{\lambda(t)} &= i\lambda(t) - H_Q \\ &= i\lambda + C_Q q - [P - C - W + \lambda]q_Q. \end{aligned} \tag{19}$$

Accordingly, we obtain the $\lambda(t) = \int_t^T ((P - C - W + \lambda)q_Q - C_Q q)\, e^{-i(s-t)} ds$. Let $A = k_1 \left(k_2 + \int_{r_l}^{r_0} Lg(r)dr + \int_{r_0}^{r_u} \frac{M}{r} g(r)dr\right)^{\psi_1} P^{-\psi_2}$ and $B = K(A_2 + B_2\theta^{-\gamma_1})$. We have $q_Q = \frac{A}{Q_M}\left(1 - \psi - \frac{2Q(t)}{Q_M}\right)$ and $C_Q = -\gamma_2 B {Q_0}^{\gamma_2} Q(t)^{-\gamma_2 - 1}$. For convenience, we use a dot above a variable to denote the first derivative with respect to time. A subscript on a variable denotes partial differential with respect to that variable. For example, $\dot{P}$ is the time derivative of P, and W_L is the age limit L derivative of expected warranty cost W.

For the existence of an optimal solution, the following necessary conditions must be satisfied. The necessary conditions for the optimal solution are given in Equation 20.

$$\begin{cases} H_P = 0 \Rightarrow P - C - W + \lambda = -\dfrac{q}{q_P} \\ H_L = 0 \Rightarrow P - C - W + \lambda = \dfrac{W_L q}{q_L} \end{cases}. \tag{20}$$

We can obtain $W_L = -\frac{q_L}{q_P}$ from Equation 20.

In addition, the sufficient conditions for the existence of the optimal solution are given by the second-order conditions for H-maximization, which are given as follows.

$$\begin{cases} H_{PP} < 0 \Rightarrow 2q_P - (\frac{q}{q_P})q_{PP} < 0 \\ H_{LL} < 0 \Rightarrow -W_{LL}q - 2W_L q_L + (\frac{W_L q}{q_L})q_{LL} < 0 \\ H_{PP}H_{LL} - H_{PL}^2 > 0 \end{cases} \cdot \tag{21}$$

where

$$H_{PL} = H_{LP} = -(\frac{q}{q_P})q_{PL} + q_L - W_L q_P. \tag{22}$$

Accordingly, $HM = \begin{bmatrix} H_{PP} & H_{PL} \\ H_{LP} & H_{LL} \end{bmatrix}$ is a negative definite matrix.

We derive the following lemmas.

Lemma 4.1. *If $H_{LL} + W_L H_{PL} > 0$, then $H_{PL} > 0$ and $H_{PL} + W_L H_{PP} < 0$.*

Lemma 4.2. *If $H_{PL} + W_L H_{PP} > 0$, then $H_{LP} > 0$ and $H_{LL} + W_L H_{PL} < 0$.*

According to Lemma 4.1 and Lemma 4.2, we have $H_{PL} > 0$ and $H_{LP} > 0$. It indicates that the total profit $H(t)$ at time t will be increased by increasing or decreasing both sale price and 2D warranty period simultaneously. In this case, the manufacturer adopts a strategy that sale price and 2D warranty period increase or decrease simultaneously to increase the total expected profit.

As in Equation 11, the sales rate function $q(t)$ can be expressed as follows.

$$q(t) = F(P(t), L(t))G(Q(t)), \tag{23}$$

where

$$F(P(t), L(t)) = k_1 \left[k_2 + \int_{r_l}^{r_0} L(t)g(r)dr + \int_{r_0}^{r_u} \frac{M(t)}{r} g(r)dr \right]^{\psi_1} P(t)^{-\psi_2},$$

and

$$G(Q(t)) = \left[1 - \frac{Q(t)}{Q_M} \right] \left[\psi + \frac{Q(t)}{Q_M} \right].$$

Accordingly, the following equations can be obtained.

$$\begin{cases} H_{PP} = \dfrac{G}{F_P}(2F_P^2 - F_{PP}F) \\ H_{LL} = \dfrac{F_L G}{F_P^2}(2F_L F_P - F_{PL}F) \\ H_{PL} = H_{LP} = \dfrac{G}{F_P}(2F_L F_P - F_{PL}F) \end{cases}, \tag{24}$$

and $W_L = -\frac{q_L}{q_P} = -\frac{F_L}{F_P}$.

To analyze the relationship between optimal sale price and warranty period, we take time derivative of sale price and warranty period in Equation 20,respectively. Then, substitute Equation 19 for $\dot{\lambda}(t)$ and W_L. Thus, the optimal solution is given as the following equations.

$$\begin{cases} H_{PP}\dot{P} + H_{PL}\dot{L} = (-i\lambda F_P - F^2 G_Q)G \\ H_{LP}\dot{P} + H_{LL}\dot{L} = (-i\lambda F_P - F^2 G_Q)G\dfrac{F_L}{F_P} \end{cases}. \tag{25}$$

Accordingly, the optimal $\dot{P}$ and $\dot{L}$ is given as follows.

$$\begin{cases} \dot{P} = \dfrac{H_{LL} + W_L H_{PL}}{H_{PP}H_{LL} - H_{PL}^2}(-i\lambda F_P - F^2 G_Q)G \\ \dot{L} = \dfrac{H_{PL} + W_L H_{PP}}{H_{PL}^2 - H_{PP}H_{LL}}(-i\lambda F_P - F^2 G_Q)G \end{cases}. \tag{26}$$

Table 3: Optimal policies for a zero discount rate $i = 0$

Condition	Diffusion effect $G_Q > 0$	Saturation effect $G_Q < 0$
$H_{LL} + W_L H_{PL} > 0$	$\dot{P} < 0, \dot{L} < 0$	$\dot{P} > 0, \dot{L} > 0$
$H_{PL} + W_L H_{PP} > 0$	$\dot{P} > 0, \dot{L} > 0$	$\dot{P} < 0, \dot{L} < 0$

In the case of a zero discount rate $i = 0$, the obtained results are summarized in Table 3. When the discount rate i is relatively low, it can be used as approximation optimal results. Let $\Delta = k_1 \left(k_2 + \int_{r_l}^{r_0} Lg(r)dr + \int_{r_0}^{r_u} \frac{M}{r} g(r)dr\right)^{\psi_1}$. Based on the Lammas 4.1 and 4.2 and Equation 26, the optimal strategies are summarized in Table 4, when the discount rate $i > 0$.

Table 4: Optimal policies for a positive discount rate $i > 0$

Condition	$i\lambda\psi_2 < \Delta P^{(1-\psi_2)} \frac{Q_M(1-\psi)-2Q}{Q_M^2}$	$i\lambda\psi_2 > \Delta P^{(1-\psi_2)} \frac{Q_M(1-\psi)-2Q}{Q_M^2}$
$H_{LL} + W_L H_{PL} > 0$	$\dot{P} < 0, \dot{L} < 0$	$\dot{P} > 0, \dot{L} > 0$
$H_{PL} + W_L H_{PP} > 0$	$\dot{P} > 0, \dot{L} > 0$	$\dot{P} < 0, \dot{L} < 0$

5. Case study

We consider a case study of industrial 3D printer in China, and the manufacturer provides a 2D FRW warranty policy $\Omega(L, M)$ to the customer. The degradation of the 3D printer is affected by both age and usage, and the usage is measured by the number of prints. The product life cycle in the market is $T = 5$ years. The hazard function of products with usage rate r is $h(t|r,\theta) = \theta^\beta \beta t^{\beta-1} r^{\eta\beta}$, where θ is the design parameter with interval $[0.05, 0.2]$ and the shape parameter is $\beta = 2$. $\eta = 0.2$ is an accelerated factor parameter for different usage rates. A smaller value of θ represents a smaller hazard rate of the product.

The total design and development cost $C_d(\theta)$ is given as $C_d(\theta) = 50000+50000\exp(\frac{0.5(0.2-\theta)}{\theta-0.05})$. Considering the learning effects during manufacturing, the production cost per unit at time t is $C_m(t,\theta) = (10+30\theta^{-0.5}(2000/Q(t))^{0.4}$. All failures during warranty period are minimally repaired by the manufacturer with an average repair cost $C_r = 30$. The usage rates of customers are different during warranty period. According to maintenance data, the probability density function of R follows a log-normal distribution as shown in Equation 27.

$$g(r) = \frac{1}{r\sigma\sqrt{2\pi}} \exp\left(\frac{-(\ln r - \mu)^2}{2\sigma^2}\right), \tag{27}$$

where $\mu = 0.5$ and $\sigma = 0.5$. Based on historical sales data, the maximum sales potential in the market is $Q_M = 25000$. The estimated parameters of the sale rate function are $k_1 = 10^8$,$k_2 = 0.1$,$\psi = 0.5$,$\psi_1 = 0.1$, and $\psi_2 = 2$. The annual discounting rate is $i = 0.15$. In order to maximize total expected profit, the manufacturer is to determine the optimal design of reliability parameter, 2D warranty period and sale price simultaneously.

Table 5: The results for the two scenarios with $T = 5$

Scenario	Period	P^*	L^*	M^*	r_0^*	θ^*	$Q(T)$	J^*
Scenario 1	*1-5*	**85.52**	**1.31**	**2.39**	**1.82**	**0.20**	11386	7.1825×10^5
	1	**143.51**	**2.48**	**4.52**				
	2	**113.09**	**2.20**	**4.00**				
Scenario 2	*3*	**95.13**	**2.03**	**3.70**	**1.82**	**0.14**	21554	6.8893×10^5
	4	**88.93**	**1.94**	**3.53**				
	5	**84.80**	**1.89**	**3.45**				

5.1. A stable market condition (scenario 1)

In a stable market condition, the sale price and the 2D warranty period stay constant during the life cycle in the market. When a stable market condition is considered, the optimal sale price is $P^* = 85.52$ and the optimal 2D warranty period is $\Omega(1.31, 2.39)$, as shown in Table 5. In this case, the optimal design reliability parameter is 0.2, which is the minimum reliability requirements for satisfactory operation.

Table 6: The results for different values β (scenario 1,$T = 5$)

β	P^*	L^*	M^*	r_0^*	θ^*	$Q(T)$	J^*
1.5	**87.43**	**1.28**	**1.48**	**1.16**	**0.20**	11323	6.8921×10^5
2.0	**85.52**	**1.31**	**2.39**	**1.82**	**0.20**	11386	7.1825×10^5
2.5	**84.37**	**1.50**	**3.20**	**2.13**	**0.20**	11424	7.3858×10^5

We investigate the effects of shape parameter β and the accelerated factor parameter η on the optimal values. A larger value β means a larger failure rate and a lower product's reliability. As shown in Table 6, for a stable market condition, the manufacturer will conduct a lower sale price and a longer 2D warranty period strategy when the product's reliability is lower. This is because the manufacturer can obtain much more benefits of increasing products' sales quantity due to lower price and longer warranty period than the increased

Table 7: The results for different values η (scenario 1,$T = 5$)

η	P^*	L^*	M^*	r_0^*	θ^*	$Q(T)$	J^*
0.1	**85.51**	**1.24**	**3.99**	**3.21**	**0.20**	11386	7.2165×10^5
0.2	**85.52**	**1.31**	**2.39**	**1.82**	**0.20**	11386	7.1825×10^5
0.4	**85.53**	**2.44**	**1.61**	**0.66**	**0.20**	11385	7.1363×10^5

warranty cost. The effects of the accelerated factor for usage rate η on the optimal values are also considered. A larger parameter η leads to a higher failure rate of products with usage rate r. As shown in Table 7, the optimal nominal usage rate of 2D warranty period decreases with the increasing of parameter η due to a higher product's failure rate. This's because for a larger η a longer age limit of 2D warranty period is offered to promote sale, while a lower usage limit of 2D warranty period is preferred to reduce the expected warranty cost.

Table 8: The results for different values μ (scenario 1,$T = 5$)

μ	P^*	L^*	M^*	r_0^*	θ^*	$Q(T)$	J^*
0.3	**85.52**	**1.37**	**2.04**	**1.49**	**0.20**	11384	7.2112×10^5
0.5	**85.52**	**1.31**	**2.39**	**1.82**	**0.20**	11386	7.1825×10^5
0.7	**85.53**	**1.26**	**2.81**	**2.22**	**0.20**	11385	7.1541×10^5

Table 9: The results for different values σ (scenario 1,$T = 5$)

σ	P^*	L^*	M^*	r_0^*	θ^*	$Q(T)$	J^*
0.3	**85.52**	**1.24**	**2.24**	**1.80**	**0.20**	11386	7.1820×10^5
0.5	**85.52**	**1.31**	**2.39**	**1.82**	**0.20**	11386	7.1825×10^5
0.7	**85.52**	**1.40**	**2.48**	**1.77**	**0.20**	11386	7.1838×10^5

When the usage rate of the customers changes, their effects on the optimal values are discussed as follows. In this case study, usage rate of the customers follows a log-normal distribution with parameter μ and σ. Parameter μ represents the mean value of $\log(r)$. As shown in Table 8, when customers' mean value of $\log(r)$ increases, the optimal nominal usage rate of the 2D warranty period increases. The optimal age limit L^* decreases and the optimal cumulative usage limit M^* increases when the parameter μ increases. While the optimal sale price changes slightly. It implies that parameter μ mainly influences the optimal values of 2D warranty period. The parameter σ represents the variance among customers' usage rate. When the variance increases, as shown in Table 9, both the optimal age limit L^* and the cumulative usage limit M^* increases. Similar to parameter μ, the optimal sale price changes slightly. This's because when the variance of customers' usage rate increases, the expected warranty cost for a same 2D warranty period will decrease, and a longer 2D warranty period can be offered to promote sales quantity.

5.2. A dynamic market condition (scenario 2)

When a dynamic market condition is considered, the manufacturer changes sale price and 2D warranty period once a year. When the life cycle $T = 5$, we have sales price $P_1, P_2, ...P_5$ and 2D warranty period $\Omega(L_1, M_1), \Omega(L_2, M_2), ..., \Omega(L_5, M_5)$, and one design reliability parameter θ. A two-stage dynamic optimization method which has been proposed by Darghouth et al. (2017) is used to find the optimal values under scenario 2. The above procedure is repeated for different values of θ until the optimal values of sale price and 2D warranty period are obtained. As shown in Tabel 5, the optimal design reliability parameter is 0.14. Both the optimal sale price and 2D warranty period are decreasing during the life cycle. An interesting finding of scenario 2 is that the optimal nominal usage rate $r_0^* = 1.82$ is constant during the life cycle. It implies that the manufacturer provides the optimal 2D warranty period with a constant nominal usage rate during the life cycle, which is also very convenient for the manufacturer to conduct the strategy in practice.

For a dynamic market condition, the optimal sale price $P_1^*, P_2^*, ..., P_5^*$ and the 2D warranty period deceases over time for a certain value of β, as shown in Table 10. It means that the manufacturer conducts a low price strategy to promote sales over the life cycle, while offering

Table 10: The results for different values β (scenario 2)

β	Period	P^*	L^*	M^*	r_0^*	θ^*	$Q(T)$	J^*
	1	**140.71**	**2.42**	**2.81**				
	2	**110.76**	**2.05**	**2.38**				
1.5	*3*	**97.11**	**1.84**	**2.14**	**1.16**	**0.15**	21622	6.7969×10^5
	4	**87.18**	**1.74**	**2.01**				
	5	**83.17**	**1.68**	**1.94**				
	1	**143.51**	**2.48**	**4.52**				
	2	**113.09**	**2.20**	**4.00**				
2.0	*3*	**95.13**	**2.03**	**3.70**	**1.82**	**0.14**	21554	6.8893×10^5
	4	**88.93**	**1.94**	**3.53**				
	5	**84.80**	**1.89**	**3.45**				
	1	**149.0**	**2.94**	**6.27**				
	2	**118.0**	**2.68**	**5.70**				
2.5	*3*	**101.0**	**2.52**	**5.35**	**2.13**	**0.13**	21153	6.9509×10^5
	4	**92.6**	**2.42**	**5.16**				
	5	**88.0**	**2.37**	**5.05**				

Table 11: The results for different values η (scenario 2)

η	Period	P^*	L^*	M^*	r_0^*	θ^*	$Q(T)$	J^*
	1	**143.0**	**2.33**	**7.50**				
	2	**113.0**	**2.07**	**6.64**				
0.1	*3*	**96.9**	**1.91**	**6.14**	**3.21**	**0.14**	21609	6.9069×10^5
	4	**88.8**	**1.83**	**5.87**				
	5	**84.7**	**1.78**	**5.73**				
	1	**143.51**	**2.48**	**4.52**				
	2	**113.09**	**2.20**	**4.00**				
0.2	*3*	**95.13**	**2.03**	**3.70**	**1.82**	**0.14**	21554	6.8893×10^5
	4	**88.93**	**1.94**	**3.53**				
	5	**84.80**	**1.89**	**3.45**				
	1	**144.0**	**4.61**	**3.04**				
	2	**113.0**	**4.08**	**2.69**				
0.4	*3*	**97.4**	**3.77**	**2.48**	**0.66**	**0.14**	21478	6.8649×10^5
	4	**89.1**	**3.60**	**2.37**				
	5	**84.9**	**3.51**	**2.32**				

a lower warranty period to decrease the expected warranty cost for a given β. According to the results in Table 6 and 10, we note that an increasing of β also leads to an increase of the optimal nominal usage rate r_0^* for the two scenarios. In the case of β is higher, the manufacturer provides a longer cumulative usage limit of 2D warranty region to promote sales, while the age limit of 2D warranty region is controlled under certian value to decease the expected warranty cost. Meanwhile, β influences the optimal reliability parameter θ^* in the design stage. Specifically, when a larger β is offered, the optimal reliability parameter θ^* should be smaller to provide a better design reliability.

The effects of accelerated factor for usage rate η on the optimal values are shown in Table 11. For a given η, the optimal price and warranty period decrease over time, while the optimal nominal rate is constant in life cycle. As shown in Table 7 and Table 11, we conclude that the optimal nominal rate decreases when the accelerated factor η increases for two scenarios. Unlike the cases of β in Table 6 and 10, the accelerated factor η mainly influences the usage limit of 2D warranty region. Specifically, in the case of a higher value of η, the manufacturer provides a longer age limit of 2D warranty region to promote sales, while the usage limit of 2D warranty region is shorter.

The effects of customers' usage rate parameters on the optimal warranty-price-reliability combination in the dynamic market are shown in Table 12 and Table 13. As shown in Table 12, for a given μ, both the optimal price and the optimal warranty period decrease over time. We can observe that the optimal nominal usage rate of the 2D warranty period increases when μ increases for the two scenarios. This is because when the customers' usage rate distribution changes, the perceptions of heterogeneous usage rate customers on 2D warranty region are different, which influences the sales. Specifically, when the mean value of $\log(r)$ increases, a larger value of cumulative usage limit of 2D warranty region is preferred, while the age limit of 2D warranty region is controlled to decrease the expected warranty cost. The parameter σ represents the variance among customers' usage rate. Similar to the cases of μ, the optimal price and the optimal warranty period decrease over time for a given σ, and the optimal nominal usage rate stays constant. As shown in Table 9 and Table 13, we conclude

Table 12: The results for different values μ (scenario 2)

μ	Period	P^*	L^*	M^*	r_0^*	θ^*	$Q(T)$	J^*
	1	**143.0**	**2.58**	**3.85**				
	2	**113.0**	**2.28**	**3.41**				
0.3	*3*	**96.9**	**2.11**	**3.15**	**1.49**	**0.14**	21600	6.9042×10^5
	4	**88.8**	**2.02**	**3.01**				
	5	**84.7**	**1.97**	**2.94**				
	1	**143.51**	**2.48**	**4.52**				
	2	**113.09**	**2.20**	**4.00**				
0.5	*3*	**95.13**	**2.03**	**3.70**	**1.82**	**0.14**	21554	6.8893×10^5
	4	**88.93**	**1.94**	**3.53**				
	5	**84.80**	**1.89**	**3.45**				
	1	**144.0**	**2.38**	**5.30**				
	2	**113.0**	**2.11**	**4.69**				
0.7	*3*	**97.3**	**1.95**	**4.34**	**2.22**	**0.14**	21507	6.8744×10^5
	4	**89.1**	**1.86**	**4.15**				
	5	**84.9**	**1.82**	**4.04**				

Table 13: The results for different values σ (scenario 2)

σ	Period	P^*	L^*	M^*	r_0^*	θ^*	$Q(T)$	J^*
	1	**144.0**	**2.35**	**4.23**				
	2	**113.0**	**2.08**	**3.74**				
0.3	*3*	**97.1**	**1.92**	**3.46**	**1.80**	**0.14**	21553	6.8890×10^5
	4	**88.9**	**1.84**	**3.31**				
	5	**84.8**	**1.79**	**3.23**				
	1	**143.51**	**2.48**	**4.52**				
	2	**113.09**	**2.20**	**4.00**				
0.5	*3*	**95.13**	**2.03**	**3.70**	**1.82**	**0.14**	21554	6.8893×10^5
	4	**88.93**	**1.94**	**3.53**				
	5	**84.80**	**1.89**	**3.45**				
	1	**144.0**	**2.64**	**4.67**				
	2	**113.0**	**2.34**	**4.14**				
0.7	*3*	**97.1**	**2.16**	**3.83**	**1.77**	**0.14**	21556	6.8900×10^5
	4	**88.9**	**2.06**	**3.66**				
	5	**84.8**	**2.01**	**3.57**				

that when the variance of customers' usage rate increases, the optimal nominal usage rate of 2D warranty region decreases. Specifically, both the optimal age limit L^* and the cumulative usage limit M^* increase for the two scenarios, when the variance of customers' usage rate increases. This is because when the variance of customers' usage rate increases, the expected warranty cost for a same 2D warranty period will decrease, and a longer 2D warranty period can be offered to promote sales quantity.

5.3. Management implications

According to the above results, the following management implications are presented. For 2D warranted products with heterogeneouss usage rate, these implications can help the manufacturer to make the optimal decision to conduct a warranty-reliability-price combination strategy.

Firstly, the optimal price and 2D warranty period decrease or increase simultaneously in the dynamic market condition, as given in Table 3 and Table 4. For example, when the market is in the diffusion stage as described by Bass (1969), the manufacturer conducts a decreasing pricing strategy to promote sales quantity, and a decreasing 2D warranty period strategy to decrease expected warranty cost.

Meanwhile, the effects of usage rate on the optimal price, 2D warranty period and the design reliability parameter are different. Specifically, both the mean value μ and the variance σ among customers' usage rate influences the optimal price and the design reliability parameter slightly. When the mean value μ of usage rate increases, the optimal age limit of 2D warranty period decreases, while the optimal usage limit increase, as shown in Table 8 and Table 12. The optimal nominal usage rate of 2D warranty period increases with the increasing of the mean value μ of usage rate. However, when the variance σ among customers' usage rate increases, both the optimal age limit and the usage limit of 2D warranty period increase, as shown in Table 9 and Table 13. While the optimal nominal usage rate of 2D warranty period changes slightly.

Moreover, the optimal nominal usage rate of 2D warranty period stays constant in a dynamic market condition. It's a practicable strategy for the manufacturer to conduct the

2D warranty period with a constant nominal usage rate. The optimal nominal usage rate is influenced by many factors, such as reliability related paramaters (e.g. the accelerated factor parameter η) and customers' usage rate distribution (e.g. the mean value of $\log(r)$). For example, when the mean value of customers' usage rate increases, the optimal nominal usage rate will increase accordingly.

Lastly, the optimal design reliability of conducting a dynamic market strategy is greater than that of conducting a stable market strategy. It's because that the optimal 2D warranty period in a dynamic market condition is usually longer than that in a stable market strategy. It means that a better design reliability is required in a dynamic market condition to decrease failures during warranty period and the related expected warranty cost. In this paper, a smaller value of design reliability parameter θ^* is obtained in the case of a dynamic market condition. As described in Table 10 - 13, the optimal design reliability parameter θ^* in the dynamic market condition is mainly influenced by the shape parameter β. For example, when the shape parameter β increases, the optimal θ^* decreases.

6. Conclusions and future works

To reduce the risks in the after-sales service contracts, the manufacturer should consider the warranty period, sale price and product's reliability jointly. For complex repairable products with a 2D Free Replacement-repair Warranty, a systematic warranty-reliability-price combination decision model was proposed. Customers are heterogeneous on usage rates during the 2D warranty region, which influences both the reliability modeling and the demand function of 2D warranted products. The sales rate function of 2D warranted products considering customers' different usage rates was proposed, and the effects of adoption and diffusion were modelled based on the Bass's growth model. Two marketing scenarios, i.e., a stable market and a dynamic market, were investigated to maximizing total expected discounted profits over product's life cycle. The optimality conditions for the two scenarios were discussed. A case of an industrial 3D printer manufacturer was illustrated to show the applicability of the proposed model, and sensitivity analysis of key parameters were conducted. We found that the optimal nominal usage rate of 2D warranty period stays constant, and the optimal price and 2D warranty period decrease or increase simultaneously in the dynamic market condition. It's a practicable strategy for the manufacturer to conduct the dynamic 2D warranty period and pricing. Meanwhile, when considering the design reliability jointly with price and warranty, we found that the expected profit in the static market was higher than that in the dynamic market, but the market share of a dynamic market condition was higher than that of a stable market condition. That's because the optimal design reliability of conducting a dynamic market strategy was greater than that of conducting a stable market strategy. The proposed model can help the manufacturer to make the optimal decision to conduct a 2D warranty-reliability-price combination strategy.

Other than the works have been done in this study, there are still some topics worth paying attention to in the future. The effects of preventive maintenance during 2D warranty region is not considered in this paper, which influences the field reliability and the expected warranty cost. Another extension is to consider the warranty-reliability-price combination issue in the context of other types of two-dimensional warranty policies, such as renewing FRW warranty or pro-rata warranty policies.

Disclosure

No potential conflict of interest was reported by the authors.

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